Embeddings & Immersions of Manifolds: Whitney-Stiefel classes & Smale's Theorem

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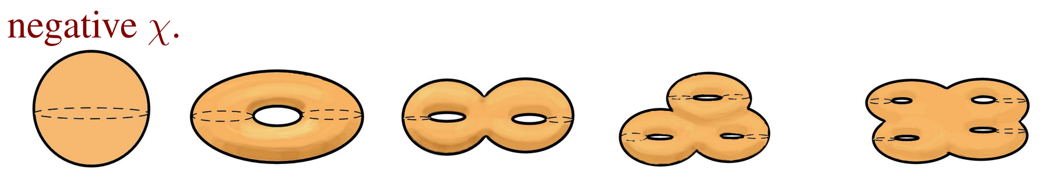
Introduction

Differential Topology is the study of smooth manifolds and their differentiable structures. In this project I will present some of the most oustanding and surprising results obtained in the field over the last fifty years mostly on the problem of embedding and immersions of manifolds. In order to do so I will introduce some technical device including vector bundles and Stiffel-Whitney (Chern-Pontryagin) classes that serve as key ingredients in formulating necessary and sufficient conditions for such immersions to exist. As a remarakble consequence I will present the extraordinarily surprising theorem of Steven Smale (1954) on eversion of spheres in three space which astounded the mathematical community for decades!

Classification of Surfaces, Genus & Euler Characteristic

Let us start by looking at *surfaces*, that is, 2-dimensional smooth closed manifolds. Then by a classical result in topology each such surface is diffeomorphic to a sphere with g handles attached to it. The number g here is a topological invariant and is called the *genus* of the surface. A related and equally useful notion is that of Euler characteristic χ defined as $\chi = 2 - 2g$. So the sphere ($g = 0$) has $\chi = 2$ and the torus ($g = 1$) has $\chi = 0$ and all other surfaces have negative χ.



g=0 g=1 g=2 g=3 g=4

Embeddings and Immersions of Manifolds

An immersion is a mapping of one smooth manifold into another whose differential satisfies a certain non-degeneracy condition. An embedding is an immersion which is additionally injective. This can be easily seen in the case of the circle in the plane. Any smooth closed curve in the plane is an immersion of the circle where as the only embeddings of the circle are smooth Jordan curves! One of the fundamental problems in differential topology is to characterise, for a given pair of manifolds, all possible classes of immersion of one manifold into the other. An indespensible tool for doing this are the so-called "Characteristic classes" described below.

Some Classes of n-Manifolds

For the sake of clarity here we list some important classes of manifolds that frequently occur in the theory:

1. Sphere $\mathbb{S}^n$.
2. Projective spaces:
   (a) Real $\mathbb{P}_n(\mathbb{R})$, (b) Complex $\mathbb{P}_n(\mathbb{C})$, (c) Quaternionic $\mathbb{P}_n(\mathbb{H})$.
3. Grassmann and Stiefel manifolds $G_{n,k}, V_{n,k}$.
4. Orthogonal and Special Orthogonal Groups $\mathbf{O}(n), \mathbf{SO}(n)$.
5. Unitary and Special Unitary Groups $\mathbf{U}(n), \mathbf{SU}(n)$.

Vector Bundles

A vector bundle over a manifold is an assignment of a vector spaces (real or complex) to each point of the manifold. Whilst locally the structure of a vector bundle is dictated by the structure of the vector space the picture is completely different globally. The study of vector bundles over a manifolds says a lot about the topology and immersions of the manifold. We proceed by first presenting the precise definition leaving the discussion and some prominent exmaples of vector bundles to the next section. A real vector bundle ξ over a base space B consists of the following:

1. A topological space E=E(ξ) reffered to as the total space.
2. A projection map π:E→B .
3. the structure of a vector space $\forall$ b∈B over the real numbers in the set $\pi^{-1}(b)$.

Note that it is the last condition above that describes the local structure of a bundle as that of its corresponding vector space.

The Tangent Bundle

The *Tangent bundle* τ_M of a manifold M is a vector bundle in which the total space DM is formed of the pairs (x, v) with $x \in M$ and v in the tangent space to M at x. The projection map $\pi : DM \to M$ such that $\pi(x, v) = x$ and the vector space structure $\pi^{-1}(x)$ defined by $t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2)$

The Normal Bundle

The *Normal bundle* ν of a manifold $M \subset \mathbb{R}^n$ is the vector bundle where the total space $E \subset Mx\mathbb{R}^n$ is formed of the pairs (x, v) where v is orthogonal to the tangent space of M at x. The projection map $\pi : E \to M$. The vector space structore in $\pi^{-1}(x)$ defined by $t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2)$

Whitney-Stiefel Characteristic Classes

In general a Characteristic class is a cohomology class associated to a vector bundle attached to a topological space. What concerns us most in this research, and the problem of immerssions of manifolds, are primarily the *Stiefel-Whitney* and the *Chern-Pontryagin* classes. To put this into context we present the following fundamental existence result on the Stiefel-Whitney class: There is a cohomology class $w_i(\xi)$ on each vector bundle ξ of a manifold where $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2), i = 0, 1, 2, ..., H^i(B\xi); \mathbb{Z}/2)$ is the i^{th} singular cohomology groups of B with coefficients in $\mathcal{F}/2$ $w_i(\xi)$ is the Stiefel-Whitney class of ξ $w_i(\xi)$ satisfies the following

1. If a bundle map covers f:B(ξ) $\to B(\eta)$ then : $w_i(\xi) = f^*(w_i(\eta))$.
2. If ξ and η are vector bundles over the same base space, then $w_k(\xi + \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)$.

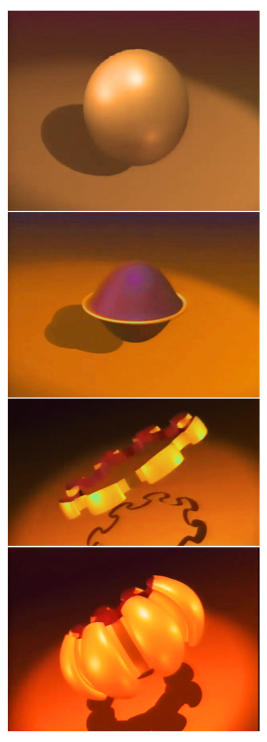
One can think of these Characteristic classes as obstruction cocyles asscociated with the extendibility of maps from the manifolds and its corresponding vector bundle to the Stiefel manifold $V_{n,k}$.In what follows we show how this device can be used to solve the eversion problem for the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

The n-Sphere Inversion Problem

Take a circle, try to invert it inside out without leaving the plane. This challenging task turns out to be impossible! For many years it was believed that the same is true for the 2-sphere. However much to the surprise of the mathematical world, Steven Smale, using tools from differential topology proved that it is possible to invert a 2-Sphere inside out in the 3-space. Technically speaking this means that there exists a homotopy within the class of immersions of the 2-Sphere in $\mathbb{R}^3$, starting from the identity and terminating at the antipodal map.

Using similar techniques Smale managed to give a complete proof of the Poincaré conjecture in dimensions $n \geq 5$. More precisely: If M^n is a differentiable homotopy sphere of dimension $n \geq 5$, then M^n is homeomorphic to S^n. In fact, M^n is diffeomorphic to a manifold obtained by gluing together the boundaries of two closed n-balls under a suitable diffeomorphism.

Method

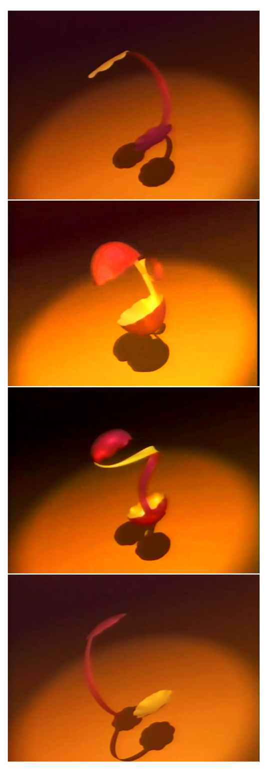


Consider a Sphere which can be bent, and stretched, and pass through itself. But we can not make tight creases.

We can not simply pass the sphere through itself since this creates a tight crease.

Now imagine the Sphere is made up of a series of circles which we give a wavy boundary.

We can then stretch this circle to create most of our sphere then use a dome at the top and bottom to show the poles.

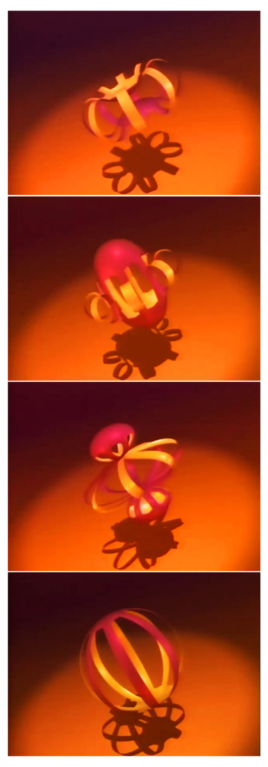


We represent one of these waves with a guide, with the poles at the top and bottom, we now want to turn this guide inside out.

To start we pass the poles through each other, but not far enough to form a crease from the loop.

Then rotate the poles once in opposite directions.

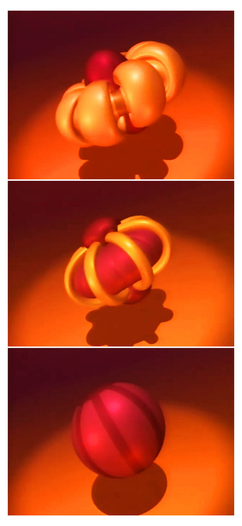
This untwists the loop and our guide has been turned inside out.



Now consider the same process with multiple guides.

Since the surface can pass through itself, each guide can be turned inside out at the same time.

We then have that the entire Sphere has been turned inside out without making any holes or tight creases.



Poincar$\acute{e}$-Hopf Theorem

Let X be a smooth vector field on a compact manifold M^n. If X has only isolated zeros then, Index(X)=$\chi(M^n)$. Here

$$\chi(M^n) = \sum_{i=0}^{n} (-1)^i \beta_i(M), \tag{1}$$

where β_i is the i-th Betti number on M^n: $\beta_i = dim_{\mathbb{R}} H^i(M^n)$. As $\chi(M^n)$ is a topological invariant of M^n then so is the index of X!

Existence of an immersion

According to Whitneys embedding theorem every smooth manifold M^n embedds smoothly in $\mathbb{R}^{2n}$ and immerses smoothly into $\mathbb{R}^{2n-1}$. The device of characterstic classes and the vanishing of the corresponding cohomology co-cyles dictates whether one can reduce the dimensions further in the target Euclidean space (e.g., if $w_i(M^m) \neq 0, i < k$ Then M can not be immersed in $\mathbb{R}^{m+k}$).

Examples and results

1. If M^n is parallelisable then it can be immersed in $\mathbb{R}^{n+1}$.
2. Every closed 3-manifold can be immersed in $\mathbb{R}^4$.
3. If $n \equiv 1(4)$ then M^n can be immersed in $\mathbb{R}^{2n-2}$.
4. $\mathbb{P}_n(\mathbb{R})$ can ***not*** be immersed in $\mathbb{R}^{2n-2}$ with $n = 2^s$.
5. $\mathbb{P}_2(\mathbb{R})$ can ***not*** be embedded in $\mathbb{R}^3$ but can in $\mathbb{R}^4$.

(Note that a manifold is said to be parallelisable *iff* its tangent bundle is trivial. As an example the only parallelisable spheres are $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ and no more!)

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